Two-Ports and Power Gain

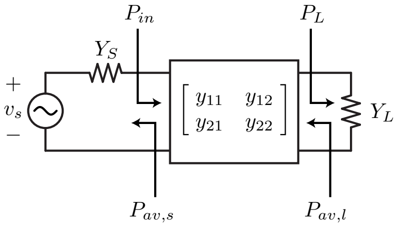
Prof. Ali M. Niknejad

U.C. Berkeley

September 28, 2017

Power Gain

Power Gain



- We can define power gain in many different ways. The *power gain* G_p is defined as follows

$$G_p = \frac{P_L}{P_{in}} = f(Y_L, Y_{ij}) \neq f(Y_S)$$

- We note that this power gain is a function of the load admittance Y_L and the two-port parameters Y_{ij}.

Power Gain (cont)

- The *available power gain* is defined as follows

$$G_a = \frac{P_{av,L}}{P_{av,S}} = f(Y_S, Y_{ij}) \neq f(Y_L)$$

- The available power from the two-port is denoted $P_{av,L}$ whereas the power available from the source is $P_{av,S}$.
- Finally, the *transducer gain* is defined by

$$G_T = \frac{P_L}{P_{av,S}} = f(Y_L, Y_S, Y_{ij})$$

- This is a measure of the efficacy of the two-port as it compares the power at the load to a simple conjugate match.

Derivation of Power Gain

- The power gain is readily calculated from the input admittance and voltage gain

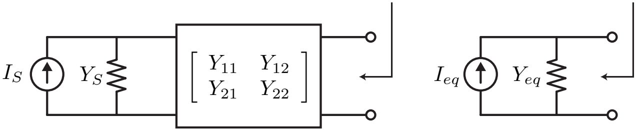
$$P_{in} = \frac{|V_1|^2}{2} \Re(Y_{in})$$

$$P_L = \frac{|V_2|^2}{2} \Re(Y_L)$$

$$G_p = \left|\frac{V_2}{V_1}\right|^2 \frac{\Re(Y_L)}{\Re(Y_{in})}$$

$$G_p = \frac{|Y_{21}|^2}{|Y_L + Y_{22}|^2} \frac{\Re(Y_L)}{\Re(Y_{in})}$$

Derivation of Available Gain



- To derive the available power gain, consider a Norton equivalent for the two-port where (short port 2)

$$I_{eq} = -I_2 = Y_{21} V_1 = \frac{-Y_{21}}{Y_{11} + Y_S} I_S$$

- The Norton equivalent admittance is simply the output admittance of the two-port

$$Y_{eq} = Y_{22} - \frac{Y_{21} Y_{12}}{Y_{11} + Y_S}$$

Available Gain (cont)

- The available power at the source and load are given by

$$P_{av,S} = \frac{|I_S|^2}{8\Re(Y_S)} \qquad\qquad P_{av,L} = \frac{|I_{eq}|^2}{8\Re(Y_{eq})}$$

$$G_a = \left|\frac{I_{eq}}{I_S}\right|^2 \frac{\Re(Y_S)}{\Re(Y_{eq})}$$

$$G_a = \left|\frac{Y_{21}}{Y_{11} + Y_S}\right|^2 \frac{\Re(Y_S)}{\Re(Y_{eq})}$$

Transducer Gain Derivation

- The transducer gain is given by

$$G_T = \frac{P_L}{P_{av,S}} = \frac{\frac{1}{2}\Re(Y_L)|V_2|^2}{\frac{|I_S|^2}{8\Re(Y_S)}} = 4\Re(Y_L)\Re(Y_S)\left|\frac{V_2}{I_S}\right|^2$$

- We need to find the output voltage in terms of the source current. Using the voltage gain we have and input admittance we have

$$\left|\frac{V_2}{V_1}\right| = \left|\frac{Y_{21}}{Y_L + Y_{22}}\right|$$

$$I_S = V_1(Y_S + Y_{in})$$

$$\left|\frac{V_2}{I_S}\right| = \left|\frac{Y_{21}}{Y_L + Y_{22}}\right| \frac{1}{|Y_S + Y_{in}|}$$

Transducer Gain (cont)

$$|Y_S + Y_{in}| = \left| Y_S + Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}} \right|$$

- We can now express the output voltage as a function of source current as

$$\left|\frac{V_2}{I_S}\right|^2 = \frac{|Y_{21}|^2}{|(Y_S + Y_{11})(Y_L + Y_{22}) - Y_{12}Y_{21}|^2}$$

- And thus the transducer gain

$$G_T = \frac{4\Re(Y_L)\Re(Y_S)|Y_{21}|^2}{|(Y_S + Y_{11})(Y_L + Y_{22}) - Y_{12}Y_{21}|^2}$$

Maximum Power Gain and the Bi-Conjugate Match

Comparison of Power Gains

- It's interesting to note that *all* of the gain expression we have derived are in the exact same form for the impedance, hybrid, and inverse hybrid matrices.
- In general, $P_L \leq P_{av,L}$, with equality for a matched load. Thus we can say that

$$G_T \leq G_a$$

- The maximum transducer gain as a function of the load impedance thus occurs when the load is conjugately matched to the two-port output impedance

$$G_{T,max,L} = \frac{P_L(Y_L = Y_{out}^*)}{P_{av,S}} = G_a$$

Comparison of Power Gains (cont)

- Likewise, since $P_{in} \leq P_{av,S}$, again with equality when the the two-port is conjugately matched to the source, we have

$$G_T \leq G_p$$

- The transducer gain is maximized with respect to the source when

$$G_{T,max,S} = G_T(Y_{in} = Y_S^*) = G_p$$

Bi-Conjugate Match

- When the input and output are simultaneously conjugately matched, or a *bi-conjugate match* has been established, we find that the transducer gain is maximized with respect to the source and load impedance

$$G_{T,max} = G_{p,max} = G_{a,max}$$

- This is thus the recipe for calculating the optimal source and load impedance in to maximize gain

$$Y_{in} = Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}} = Y_S^*$$

$$Y_{out} = Y_{22} - \frac{Y_{12}Y_{21}}{Y_S + Y_{11}} = Y_L^*$$

- Solution of the above four equations (real/imag) results in the optimal $Y_{S,opt}$ and $Y_{L,opt}$.

Calculation of Optimal Source/Load

- Another approach is to simply equate the partial derivatives of G_T with respect to the source/load admittance to find the maximum point

$$\frac{\partial G_T}{\partial G_S} = 0 \qquad \frac{\partial G_T}{\partial G_L} = 0$$

$$\frac{\partial G_T}{\partial B_S} = 0 \qquad \frac{\partial G_T}{\partial B_L} = 0$$

- Again we have four equations. But we should be smarter about this and recall that the maximum gains are all equal. Since G_a and G_p are only a function of the source or load, we can get away with only solving two equations.

Calculation of Optimal Source/Load

- Working with available gain

$$\frac{\partial G_a}{\partial G_S} = 0 \qquad \frac{\partial G_a}{\partial B_S} = 0$$

- This yields $Y_{S,opt}$ and by setting $Y_L = Y_{out}^*$ we can find the $Y_{L,opt}$.
- Likewise we can also solve

$$\frac{\partial G_p}{\partial G_L} = 0 \qquad \frac{\partial G_p}{\partial B_L} = 0$$

- And now use $Y_{S,opt} = Y_{in}^*$.

Optimal Power Gain Derivation

- Let's outline the procedure for the optimal power gain. We'll use the power gain G_p and take partials with respect to the load. Let

$$Y_{jk} = m_{jk} + jn_{jk}$$

$$Y_L = G_L + jX_L$$

$$Y_{12}Y_{21} = P + jQ = Le^{j\phi}$$

$$G_p = \frac{|Y_{21}|^2}{D} G_L$$

$$\Re\left(Y_{11} - \frac{Y_{12}Y_{21}}{Y_L + Y_{22}}\right) = m_{11} - \frac{\Re(Y_{12}Y_{21}(Y_L + Y_{22})^*)}{|Y_L + Y_{22}|^2}$$

$$D = m_{11}|Y_L + Y_{22}|^2 - P(G_L + m_{22}) - Q(B_L + n_{22})$$

$$\frac{\partial G_p}{\partial B_L} = 0 = -\frac{|Y_{21}|^2 G_L}{D^2}\frac{\partial D}{\partial B_L}$$

Optimal Load (cont)

- Solving the above equation we arrive at the following solution

$$B_{L,opt} = \frac{Q}{2m_{11}} - n_{22}$$

- In a similar fashion, solving for the optimal load conductance

$$G_{L,opt} = \frac{1}{2m_{11}}\sqrt{(2m_{11}m_{22} - P)^2 - L^2}$$

- If we substitute these values into the equation for G_p (lot's of algebra ...), we arrive at

$$G_{p,max} = \frac{|Y_{21}|^2}{2m_{11}m_{22} - P + \sqrt{(2m_{11}m_{22} - P)^2 - L^2}}$$

Final Solution

- Notice that for the solution to exists, G_L must be a real number. In other words

$$(2m_{11}m_{22} - P)^2 > L^2$$

$$(2m_{11}m_{22} - P) > L$$

$$K = \frac{2m_{11}m_{22} - P}{L} > 1$$

- This factor K plays an important role as we shall show that it also corresponds to an unconditionally stable two-port. We can recast all of the work up to here in terms of K

$$Y_{S,opt} = -j\Im(Y_{11}) + \frac{Y_{12}Y_{21} - 2\Re(Y_{11})\Re(Y_{22}) + |Y_{12}Y_{21}|(K + \sqrt{K^2 - 1})}{2\Re(Y_{22})}$$

$$Y_{L,opt} = -j\Im(Y_{22}) + \frac{Y_{12}Y_{21} - 2\Re(Y_{11})\Re(Y_{22}) + |Y_{12}Y_{21}|(K + \sqrt{K^2 - 1})}{2\Re(Y_{11})}$$

$$G_{p,max} = G_{T,max} = G_{a,max} = \frac{Y_{21}}{Y_{12}} \frac{1}{K + \sqrt{K^2 - 1}}$$

Maximum Gain

- The maximum gain is usually written in the following insightful form

$$G_{max} = \frac{Y_{21}}{Y_{12}}(K - \sqrt{K^2 - 1})$$

- For a reciprocal network, such as a passive element, $Y_{12} = Y_{21}$ and thus the maximum gain is given by the second factor

$$G_{r,max} = K - \sqrt{K^2 - 1}$$

- Since $K > 1$, $|G_{r,max}| < 1$. The reciprocal gain factor is known as the efficiency of the reciprocal network.
- The first factor, on the other hand, is a measure of the non-reciprocity.

Unilateral Maximum Gain

- For a unilateral network, the design for maximum gain is trivial. For a bi-conjugate match

$$Y_S = Y_{11}^*$$

$$Y_L = Y_{22}^*$$

$$G_{T,max} = \frac{|Y_{21}|^2}{4m_{11}m_{22}}$$

Stability of a Two-Port

Stability of a Two-Port

- A two-port is unstable if the admittance of either port has a negative conductance for a passive termination on the second port. Under such a condtion, the two-port can oscillate.
- Consider the input admittance

$$Y_{in} = G_{in} + jB_{in} = Y_{11} - \frac{Y_{12}Y_{21}}{Y_{22} + Y_L}$$

- Using the following definitions

$$Y_{11} = g_{11} + jb_{11} \qquad\qquad Y_{12}Y_{21} = P + jQ = L\angle\phi$$

$$Y_{22} = g_{22} + jb_{22} \qquad\qquad Y_L = G_L + jB_L$$

- Now substitute real/imag parts of the above quantities into Y_{in}

$$Y_{in} = g_{11} + jb_{11} - \frac{P + jQ}{g_{22} + jb_{22} + G_L + jB_L}$$

$$= g_{11} + jb_{11} - \frac{(P + jQ)(g_{22} + G_L - j(b_{22} + B_L))}{(g_{22} + G_L)^2 + (b_{22} + B_L)^2}$$

Input Conductance

- Taking the real part, we have the input conductance

$$\Re(Y_{in}) = G_{in} = g_{11} - \frac{P(g_{22}+G_L) + Q(b_{22}+B_L)}{(g_{22}+G_L)^2 + (b_{22}+B_L)^2}$$

$$= \frac{(g_{22}+G_L)^2 + (b_{22}+B_L)^2 - \frac{P}{g_{11}}(g_{22}+G_L) - \frac{Q}{g_{11}}(b_{22}+B_L)}{D}$$

- Since $D > 0$ if $g_{11} > 0$, we can focus on the numerator. Note that $g_{11} > 0$ is a requirement since otherwise oscillations would occur for a short circuit at port 2.
- The numerator can be factored into several positive terms

$$N = (g_{22}+G_L)^2 + (b_{22}+B_L)^2 - \frac{P}{g_{11}}(g_{22}+G_L) - \frac{Q}{g_{11}}(b_{22}+B_L)$$

$$= \left(G_L + \left(g_{22} - \frac{P}{2g_{11}}\right)\right)^2 + \left(B_L + \left(b_{22} - \frac{Q}{2g_{11}}\right)\right)^2 - \frac{P^2+Q^2}{4g_{11}^2}$$

Input Conductance (cont)

- Now note that the numerator can go negative only if the first two terms are smaller than the last term. To minimize the first two terms, choose $G_L = 0$ and $B_L = -\left(b_{22} - \frac{Q}{2g_{11}}\right)$ (reactive load)

$$N_{min} = \left(g_{22} - \frac{P}{2g_{11}}\right)^2 - \frac{P^2 + Q^2}{4g_{11}^2}$$

- And thus the above must remain positive, $N_{min} > 0$, so

$$\left(g_{22} + \frac{P}{2g_{11}}\right)^2 - \frac{P^2 + Q^2}{4g_{11}^2} > 0$$

$$g_{11}g_{22} > \frac{P + L}{2} = \frac{L}{2}(1 + \cos\phi)$$

Linvill/Llewellyn Stability Factors

- Using the above equation, we define the Linvill stability factor

$$L < 2g_{11}g_{22} - P$$

$$C = \frac{L}{2g_{11}g_{22} - P} < 1$$

- The two-port is stable if $0 < C < 1$.

Stability (cont)

- It's more common to use the inverse of C as the stability measure

$$\frac{2g_{11}g_{22} - P}{L} > 1$$

- The above definition of stability is perhaps the most common

$$K = \frac{2\Re(Y_{11})\Re(Y_{22}) - \Re(Y_{12}Y_{21})}{|Y_{12}Y_{21}|} > 1$$

- The above expression is identical if we interchange ports 1/2. Thus it's the general condition for stability.
- Note that $K > 1$ is the same condition for the maximum stable gain derived earlier. The connection is now more obvious. If $K < 1$, then the maximum gain is infinity!